

Partition functions in statistical mechanics, symmetric functions, and group representations

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Partition functions for noninteracting particles are known to be symmetric functions. It is shown that powerful group-theoretical techniques can be used not only to derive these relationships, but also to significantly simplify calculation of the partition functions for particles that carry internal quantum numbers. The partition function is shown to be a sum of one or more group characters. The utility of character expansions in calculating the partition functions is explored. Several examples are given to illustrate these techniques.

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I. INTRODUCTION

Although the relationship between partition functions of noninteracting quantum systems of bosons or fermions and symmetric functions commonly encountered in the group representation theory has been known for some time (e.g., see Ref. [1]) it was recently highlighted [2] as part of an attempt to gain a deeper understanding of the foundations of the field. In particular the N -particle partition function Z_N for a noninteracting gas is a complete homogeneous symmetric function of the exponentials of the single-particle energies for bosons and an elementary symmetric function of the same for fermions (definitions and a list of some properties of these functions are given in Appendix A). Such relationships can be useful, for example, in one-dimensional fermionic systems since in one space dimension interacting fermions can be considered as noninteracting bosons [3]. Spectral equivalence of bosons and fermions in one-dimensional harmonic potentials [4,5] and previously noted recursion relations connecting partition functions with different numbers of particles [6,7] can be shown to be consequences of this identification.

The aim of this paper is first to provide a simple group-theoretical proof for Z_N being a symmetric function and then to expand this result to some cases of interacting particles and to systems with both bosons and fermions (supersymmetric systems). In Sec. II we first provide a direct combinatorial proof, then show how that proof follows from group representation theory. In that section we also show that for mixed systems of bosons and fermions partition functions become graded symmetric functions which are the characters of superalgebras. In Sec. III we show that the identification of N -particle partition functions with symmetric functions coupled with character expansion techniques significantly simplify the calculation of partition functions for particles that carry internal quantum numbers. In Sec. IV we show that it is possible to utilize these techniques to calculate the partition functions for some simple interacting systems. Finally Sec. V includes a brief discussion of our results.

II. THE RELATIONSHIP BETWEEN PARTITION FUNCTION AND GROUP CHARACTERS

One can easily write down the N -particle partition function Z_N for noninteracting particles

$$Z_N = \sum_{n_1} \sum_{n_2} \cdots \left[\prod_i x_i^{n_i} \right] \delta \left(N - \sum_j n_j \right), \quad (2.1)$$

as was written, e.g., in the treatment of pion multiplicity distributions in heavy-ion collisions [8]. Here $x_i = \exp(-\beta\epsilon_i)$, where ϵ_i are the single-particle energies and δ is a Kronecker delta constraint. For bosons $n_i = 0, 1, \dots, \infty$, and for fermions $n_i = 0, 1$. Writing the δ function as an integral

$$\delta \left(N - \sum_j n_j \right) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \exp(iN\varphi) \prod_i \exp(-in_i\varphi), \quad (2.2)$$

one can easily perform the n_i sums in Eq. (2.1) to obtain

$$Z_N = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \exp(iN\varphi) \left[\prod_i [1 + \eta x_i \exp(-i\varphi)]^\eta \right], \quad (2.3)$$

where η is -1 for bosons and $+1$ for fermions. Comparing Eq. (2.3) with the generating functions given in Appendix A, Eqs. (A1) and (A2), one immediately identifies the generating functions of the symmetric functions inside the brackets in the argument of the integral. For bosons one gets

$$Z_N = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \exp(iN\varphi) \left[\sum_M h_M(x) \exp(-iM\varphi) \right] = h_N(x), \quad (2.4)$$

i.e., the complete symmetric function in the variables $x_i = \exp(-\beta\epsilon_i)$. For fermions one gets

$$Z_N = a_N(x), \quad (2.5)$$

i.e., the elementary symmetric function in the variables $x_i = \exp(-\beta\epsilon_i)$. Hence the grand canonical partition function is

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$$Z(\mu) = \sum_N h_N(x) \exp(-\beta\mu N) \quad (2.6)$$

for bosons and

$$Z(\mu) = \sum_N a_N(x) \exp(-\beta\mu N) \quad (2.7)$$

for fermions. (In this paper we will freely switch between the product of the inverse temperature and the chemical potential, $\beta\mu$, and its analytic continuation $i\phi$).

It is perhaps easier to understand the appearance of the symmetric functions in the partition function by calculating the quantity Z_N using group characters. Group characters are the traces of the representation matrices and can be expressed in terms of symmetric functions [9]. We are interested in calculating the partition function for a system of N noninteracting particles with single-particle energies ϵ_i

$$Z_N = \text{Tr}[\delta(N - \hat{N}) \exp(-\beta\hat{H})], \quad (2.8)$$

where the number operator is

$$\hat{N} = \sum_{i=1}^m c_i^\dagger c_i \quad (2.9)$$

and the Hamiltonian is

$$\hat{H} = \sum_i^m \epsilon_i c_i^\dagger c_i. \quad (2.10)$$

In these equations c_i^\dagger and c_i are the creation and annihilation operators for either bosons or fermions.

Representations of continuous groups can be associated with the Young tableaux. Using N bosons (fermions) distributed over m states, one can construct completely symmetric (antisymmetric) irreducible representations of the group $U(m)$ associated with the Young tableaux with N boxes in a row (column). Hence the δ function in Eq. (2.8) restricts the trace to a given irreducible representation. Since the quantity $\exp(-\beta\hat{H})$ is a group element, the partition function in Eq. (2.8) is simply the group character in this representation. [Technically for $\exp(-\beta\hat{H})$ to be a group element one needs to perform a Wick rotation, i.e., analytically continue the temperature to an imaginary variable]. In the theory of group representations, one can write the character of any irreducible representation in terms of the eigenvalues of the group element in the fundamental representation, $x_i = \exp(-\beta\epsilon_i)$. (In this representation the Hamiltonian is simply an $m \times m$ matrix H_{fund} with eigenvalues ϵ_i .) The characters of the representation of $U(m)$ associated with a single row (column) Young tableaux are the complete (elementary) symmetric functions of the eigenvalues of the group element in the fundamental representation. It follows then that Z_N is the complete symmetric function in the variables $x_i = \exp(-\beta\epsilon_i)$ for bosons and the elementary symmetric function in these variables for the fermions.

Using the generating functions of the symmetric functions, Eqs. (A1) and (A2) of Appendix A, the grand canonical partition functions in Eqs. (2.6) and (2.7) can be written in the form

$$Z(\lambda) = \det[1 + \eta\lambda e^{-\beta\hat{H}}]^\eta, \quad (2.11)$$

where $\lambda = e^{-\mu}$ and η is -1 for bosons and $+1$ for fermions as before. Using the relationship

$$\det \hat{A} = \exp[\text{Tr} \log \hat{A}] \quad (2.12)$$

which is valid for any operator \hat{A} one can write Eq. (2.11) as

$$\begin{aligned} Z(\lambda) &= \sum_N Z_N \lambda^N \exp[\eta \text{Tr} \log(1 + \eta\lambda e^{-\beta\hat{H}})] \\ &= \sum_k (-1)^{k+1} \eta^k \frac{1}{k} (\text{Tr} e^{-k\beta\hat{H}}) \lambda^k. \end{aligned} \quad (2.13)$$

Equating powers of λ in both sides of the Eq. (2.13) one can easily write down the recursion relation

$$NZ_N = \sum_{k=1}^N k C_k Z_{N-k}, \quad (2.14)$$

where $C_k = (-1)^{k+1} \eta^k (1/k) (\text{Tr} e^{-k\beta\hat{H}})$. Equation (2.14) is the recursion function discussed in Ref. [6]. In the studies of multiparticle distributions the quantities C_k are known as combinants [8,10,11].

If we have a mixed system of bosons and fermions we can write the Hamiltonian to be

$$\hat{H} = \sum_{i=1}^k \epsilon_i b_i^\dagger b_i + \sum_{\alpha=1}^m \epsilon_\alpha f_\alpha^\dagger f_\alpha, \quad (2.15)$$

where the boson states are labeled by the Latin indices and the fermionic states are labeled by the Greek indices. We take b_i^\dagger and b_i (f_α^\dagger and f_α) to be the creation and annihilation operators of the bosons (fermions) and ϵ_i (ϵ_α) to be the single-particle energies. Suppose we want to calculate the partition function Z_N where the *total* number of bosons *and* fermions, $N = N_B + N_F$ is fixed. Using Eqs. (2.4) and (2.5) one can easily write the expression

$$Z_N = \sum_{n+\ell=N} h_n(x_B) a_\ell(x_F), \quad (2.16)$$

where x_B represent the variables $\exp(-\beta\epsilon_i)$ of bosons and x_F represent the variables $\exp(-\beta\epsilon_\alpha)$ of fermions. The partition function in Eq. (2.16) was introduced in Ref. [12] where it was called the ‘‘graded homogeneous symmetric function’’ of degree N in the variables $\exp(-\beta\epsilon_i)$ and $[-\exp(-\beta\epsilon_\alpha)]$. It is the character of the supergroup $U(k/m)$ [12,13] [the Hamiltonian, Eq. (2.15), is an element of the corresponding superalgebra].

III. PARTITION FUNCTION FOR PARTICLES WITH INTERNAL SYMMETRIES

Whenever a quantum gas consists of noninteracting particles with an internal symmetry then it is possible to write the grand canonical partition function and then project onto a particular representation of the Lie group associated with the symmetry in consideration. Even though this approach was first introduced in Ref. [14] in the context of statistical bootstrap models, it is in fact completely generally applicable [15]. One writes the grand canonical partition function, Z ,

$$Z = \text{Tr} \exp \left[-\beta \left(\hat{H} - \sum \mu_i Q_i \right) \right], \quad (3.1)$$

where Q_i are the conserved quantities of the system and μ_i are the chemical potentials assigned to each of the relevant conservation laws. In order to find the partition function Z_r corresponding to a given representation r of the symmetry group in consideration one simply expands the grand canonical partition function in terms of the characters χ_r of the associated Lie group:

$$Z(\mu_i) = \sum_r \frac{1}{d_r} \chi_r(\mu_i) Z_r. \quad (3.2)$$

In Eq. (3.2), d_r is the dimension of the representation. Since when the group variables are set to zero the character of a given representation gives the dimension of this representation, $\chi_r(\mu_i=0) = d_r$, when all μ_i are set to zero Eq. (3.2) gives the grand canonical partition function as a sum over all the representations as it should. In fact Eqs. (2.6) and (2.7) can be considered as special cases of Eq. (3.2) since the particle number is a conserved U(1) symmetry with characters $\exp(-iN\phi)$. Character expansions for various groups are readily available in the literature [16–18]. Such expansions of the grand canonical partition function were utilized in a variety of contexts from understanding the role of internal symmetries in $p\bar{p}$ annihilation [19] to imposing color neutrality in a quark-gluon plasma [20].

In this section we show that symmetric functions are very useful in generating such expansions. To illustrate this let us introduce two kinds of fermions which are the spin-up and spin-down components of a su(2) algebra which can be the ordinary spin, or the isospin or a pseudospin. For the purposes of fixing the notation we will call these fermions protons (with creation and annihilation operators $f_{\alpha,+}^\dagger, f_{\beta,+}$) and neutrons (with creation and annihilation operators $f_{\alpha,-}^\dagger, f_{\beta,-}$) and the symmetry isospin. We consider a dilute gas so that the interactions between these particles can be ignored and we assume that they sit at the same energy levels which may correspond to a mean field:

$$\hat{H} = \sum_{\alpha,\sigma=\pm} \epsilon_\alpha f_{\alpha,\sigma}^\dagger f_{\alpha,\sigma}. \quad (3.3)$$

It is easy to write down the generators of the corresponding su(2) algebra as

$$\hat{T}_3 = \frac{1}{2} \sum_\alpha [f_{\alpha,+}^\dagger f_{\alpha,+} - f_{\alpha,-}^\dagger f_{\alpha,-}] = \frac{1}{2} (\hat{N}_+ - \hat{N}_-), \quad (3.4)$$

and

$$\hat{T}_+ = \sum_\alpha f_{\alpha,+}^\dagger f_{\alpha,-} = (T_-)^\dagger. \quad (3.5)$$

In Eq. (3.4), \hat{N}_+ and \hat{N}_- are the number of protons and neutrons, respectively. This algebra is easy to manipulate in the $|N_+, N_- \rangle$ basis. However, if we wish to find the partition function corresponding to a particular value of isospin I , we need to go to the $|I, m_I \rangle$ basis.

Note that the third component of isospin T_3 is an additive, conserved quantum number. Using Eqs. (2.1), (2.3), (2.7) we can write the grand canonical partition function for protons as

$$\prod_\alpha (1 + x_\alpha e^{i\phi/2}) = \sum_{N_+} a_{N_+}(x) \exp(iN_+ \phi/2), \quad (3.6)$$

and for neutrons as

$$\prod_\alpha (1 + x_\alpha e^{-i\phi/2}) = \sum_{N_-} a_{N_-}(x) \exp(-iN_- \phi/2). \quad (3.7)$$

In writing Eqs. (3.6) and (3.7) we used the fact, suggested by Eq. (3.4), that the chemical potentials for protons and neutrons have opposite signs. Hence the total grand canonical partition function of the system is

$$\begin{aligned} Z &= \prod_\alpha (1 + x_\alpha e^{i\phi/2})(1 + x_\alpha e^{-i\phi/2}) \\ &= \sum_{N_+, N_-} a_{N_+} a_{N_-} \exp[i(N_+ - N_-) \phi/2]. \end{aligned} \quad (3.8)$$

On the right side of Eq. (3.8) one recognizes the matrix element of the group element $\exp(iT_3)$ in the $|N_+, N_- \rangle$ basis. One can easily write down the total grand canonical partition function as

$$Z = \prod_\alpha (1 + x_\alpha^2 + x_\alpha \text{Tr } U), \quad (3.9)$$

where U is the group element $\exp(i\phi T_3)$ in the $I = 1/2$ representation:

$$U_{I=1/2} = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}. \quad (3.10)$$

We calculate the total grand canonical partition function of Eq. (3.8) using the character expansion formula of Ref. [17], the results of which are summarized in Appendix B. Letting $t_1 = e^{i\phi/2}$ and $t_2 = e^{-i\phi/2}$ we define

$$G(x, t) \equiv \prod_{\alpha} (1 + x_{\alpha} t) = \sum_{N=0}^{\infty} a_N(x) t^N. \quad (3.11)$$

Using Eq. (B5) of Appendix B we write the total grand canonical partition function of Eq. (3.8) as

$$\left(\prod_{i=1}^2 G(x, t_i) \right) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \det(a_{n_j+i-j}) \chi_{(n_1, n_2)}(U). \quad (3.12)$$

The equations in Appendix B are given for $U(N)$. For the case of $SU(2)$ we need to write characters of a given representation I . If $n_2=0$, then $\chi_{(n_1, 0)} = \chi_{I=n_1/2}$. However, if $n_2 \neq 0$ then $\chi_{(n_1, n_2)} = \chi_{I=(n_1-n_2)/2}$. Hence the desired character expansion of the total grand canonical partition function is

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} a_n \chi_{I=n/2}(U) \\ &+ \sum_{n=0}^{\infty} \sum_{m=1}^n (a_n a_m - a_{n+1} a_{m-1}) \chi_{I=(n-m)/2}(U). \end{aligned} \quad (3.13)$$

Using Eq. (3.13) one can, for example, write down the partition function of the mixed system which corresponds to the total isospin zero as

$$Z_{I=0} = 1 + \sum_{n=1}^{\infty} (a_n^2 - a_{n+1} a_{n-1}). \quad (3.14)$$

In these equations a_n is the elementary symmetric function of degree n in the variables $x_{\alpha} = \exp(-\beta \epsilon_{\alpha})$.

To illustrate the utility of Eq. (3.14) let us consider the simple case of a one-dimensional harmonic oscillator potential for which the N -particle partition functions can be explicitly calculated (see, for example, Ref. [21]). Setting the zero-point energy of the harmonic oscillator to zero (i.e. $\hat{H} = \hbar \omega \hat{N}$) we get

$$a_N = x^{N(N-1)/2} \prod_{n=1}^N \frac{1}{(1-x^n)}, \quad (3.15)$$

where $x = \exp(-\beta \hbar \omega)$. Substituting Eq. (3.15) into Eq. (3.14) we obtain

$$Z_{I=0} = 1 + \sum_{n=1}^{\infty} a_n^2 \frac{1}{\sum_{m=0}^n x_m}, \quad (3.16)$$

where the factor $(\sum x_m)^{-1}$ projects the $I=0$ state out of a state with n proton-neutron pairs.

In the example we just considered the individual particles transformed like the fundamental representation of the internal symmetry group. We next examine what happens if they transform like another representation. Again to be specific we consider pions, which are bosons that transform like the

adjoint ($I=1$) representation of the isospin group. We assume that the energies (either the free-particle energies—i.e., we ignore the mass difference between the charged pion and the neutral pion or the mean field energies) are again the same. Introducing the creation operators $b_{i,+}^{\dagger}$, $b_{i,0}^{\dagger}$, and $b_{i,-}^{\dagger}$ for π^+ , π^0 , and π^- , respectively, the Hamiltonian is

$$\hat{H} = \sum_i \epsilon_i (b_{i,+}^{\dagger} b_{i,+} + b_{i,0}^{\dagger} b_{i,0} + b_{i,-}^{\dagger} b_{i,-}), \quad (3.17)$$

and the generators of the appropriate $SU(2)$ algebra can be written as

$$\hat{T}_+ = \sum_i (b_{i,0}^{\dagger} b_{i,-} + b_{i,+}^{\dagger} b_{i,0}) = (\hat{T}_-)^{\dagger} \quad (3.18)$$

and

$$\hat{T}_3 = \sum_i (b_{i,+}^{\dagger} b_{i,+} - b_{i,-}^{\dagger} b_{i,-}). \quad (3.19)$$

The total grand canonical partition function can again easily be written as

$$Z = \prod_i (1 - x_i e^{i\phi})^{-1} (1 - x_i)^{-1} (1 - x_i e^{-i\phi})^{-1}, \quad (3.20)$$

where $x_i = \exp(-\beta \epsilon_i)$. Note that the factor 1/2 in Eq. (3.4) is missing in Eq. (3.19) since the additive quantum number T_3 is $\pm 1/2$ for the nucleons, but ± 1 or 0 for pions. This also leads the lack of 1/2 in the exponentials multiplying the chemical potential in Eq. (3.20). Indeed the total grand canonical partition function for pions can be written as

$$Z = \det \left[\prod_i [1 - x_i U_{I=1}]^{-1} \right], \quad (3.21)$$

where the determinant is taken in the isospin space and $U_{I=1}$ is the matrix $\exp(i\phi T_3)$ in the adjoint ($I=1$) representation:

$$U_{I=1} = \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\phi} \end{pmatrix}. \quad (3.22)$$

To calculate the character expansion we proceed as before. Again using the eigenvalues of the isospin in the *fundamental* representation, $t_1 = e^{i\phi/2}$ and $t_2 = e^{-i\phi/2}$, we define

$$G(x, t) \equiv \prod_i (1 - x_i t^2)^{-1} = \sum_{N=0}^{\infty} A_N(x) t^N. \quad (3.23)$$

It follows from Eq. (A1) of Appendix A that $A_N=0$ for odd N , and $A_N = h_{N/2}$ for even N . Following the same steps as before we write

$$Z = \left[\prod_i (1 - x_i)^{-1} \right] \sum_{n_1} \sum_{n_2} \det(A_{n_j+i-j}) \chi_{(n_1, n_2)}(U). \quad (3.24)$$

Going from U(2) to SU(2) and following similar steps as those leading to Eq. (3.14) we can write down the partition function of the mixed system which corresponds to the total isospin zero as

$$Z_{I=0} = \left[\prod_i (1-x_i)^{-1} \right] \left[1 + \sum_{n=1}^{\infty} (h_n^2 - h_{n+1}) \right]. \quad (3.25)$$

One may need to calculate the total partition function of a mixed system of nucleons and pions for a particular value of isospin. To do so we can proceed in a similar way. We start with the total grand canonical partition function:

$$Z = \left[\prod_{\alpha} (1+x_{\alpha}t_1)(1+x_{\alpha}t_2) \right] \times \left[\prod_i (1-x_it_1^2)^{-1}(1-x_i)^{-1}(1-x_it_2^2)^{-1} \right]. \quad (3.26)$$

Defining the series

$$\left[\prod_{\alpha} (1+x_{\alpha}t) \right] \left[\prod_i (1-x_it^2)^{-1} \right] = \sum_N B_N(x_{\alpha}, x_i) t^N \quad (3.27)$$

we find that

$$B_N(x_{\alpha}, x_i) = \sum_{N=n+m} a_n(x_{\alpha}) \mathcal{H}_m(x_i), \quad (3.28)$$

where $\mathcal{H}_{2n}(x_i) = h_n(x_i)$ and $\mathcal{H}_{2n+1}(x_i) = 0, n = 1, 2, 3, \dots$. Using similar steps as those leading to Eq. (3.24) we get

$$Z = \left[\prod_i (1-x_i)^{-1} \right] \sum_{n_1} \sum_{n_2} \det(B_{n_j+i-j}) \chi_{(n_1, n_2)}(U). \quad (3.29)$$

It is straightforward if not tedious to generalize the discussion in this section to higher internal symmetries using the character expansion formulas in Appendix B. For SU(N), $N \geq 2$, there are $N-1$ mutually commuting operators (elements of the Cartan subalgebra). These can be expressed in terms of number operators. Each such operator is then associated with the analytic continuation of a chemical potential. From the resulting group element one follows the same procedure we just outlined.

IV. PARTITION FUNCTIONS FOR INTERACTING SYSTEMS

In some cases it is possible to utilize the techniques discussed in the previous sections to the investigation of some interacting systems. Typically if the Hamiltonian \hat{H} can be written as a sum of the generators of an algebra, then $\exp(-\beta\hat{H})$ is an element of the associated group and its trace (character) can be calculated by powerful group-theoretical methods. To illustrate this we will consider the Hamiltonian

representing a two-level system, where each level has a K -fold degeneracy, given by

$$\hat{H} = \frac{1}{2} \sum_{k=1}^K [v(f_{1,k}^{\dagger}f_{1,k} - f_{1,k}^{\dagger}f_{1,k}) + t(f_{1,k}^{\dagger}f_{2,k} + f_{2,k}^{\dagger}f_{1,k})]. \quad (4.1)$$

In Eq. (4.1), 1 and 2 represent two different layers and the degeneracy of each level is indicated by the index k . An example of such a system would be the single-particle Hamiltonian of a bilayer quantum Hall system [22]. In this case one works in a spherical geometry and the z projection of the orbital angular momentum of each electron in the lowest Landau level, k , changes from $-N_{\phi}/2$ to $N_{\phi}/2$, where N_{ϕ} is the number of flux quanta penetrating the sphere. The coefficients v and t are bias voltage and the tunneling amplitude, respectively.

To find the partition function we first diagonalize the Hamiltonian by a Bogoliubov transformation [23]:

$$F_{1,k} = (\cos \theta)f_{1,k} + (\sin \theta)f_{2,k}, \\ F_{2,k} = -(\sin \theta)f_{1,k} + (\cos \theta)f_{2,k}. \quad (4.2)$$

Note that $F_{i,k}, i = 1, 2$ such defined still satisfy the fermion anticommutation relations. In addition, under the transformation in Eq. (4.2) the total number of particles is unchanged, i.e.,

$$F_{1,k}^{\dagger}F_{2,k} + F_{2,k}^{\dagger}F_{1,k} = f_{1,k}^{\dagger}f_{1,k} + f_{2,k}^{\dagger}f_{2,k}. \quad (4.3)$$

By choosing

$$\cos \theta = \frac{v}{\sqrt{v^2 + t^2}}, \\ \sin \theta = \frac{t}{\sqrt{v^2 + t^2}}, \quad (4.4)$$

and

$$\epsilon = \sqrt{v^2 + t^2}, \quad (4.5)$$

one can write down the Hamiltonian in Eq. (4.1) in terms of the quasifermion operators:

$$\hat{H} = \frac{1}{2} \sum_{k=1}^K \epsilon [F_{1,k}^{\dagger}F_{1,k} - F_{2,k}^{\dagger}F_{2,k}]. \quad (4.6)$$

The total partition function can easily be computed as $F_{1,k}^{\dagger}$ and $F_{2,k}^{\dagger}$ create independent particles with energies $\epsilon/2$ and $-\epsilon/2$, respectively. The total possible number of both the ‘‘upper-level’’ and the ‘‘lower-level’’ particles are K . We get

$$Z = (1+x_1)^K (1+x_2)^K, \quad (4.7)$$

where $x_1 \equiv \exp(-\beta\epsilon/2)$ and $x_2 \equiv \exp(+\beta\epsilon/2)$. The partition function for a fixed number of particles can also be similarly calculated. The partition function for n ‘‘upper-level’’ par-

ticles is given by applying Eq. (2.5) to K degenerate levels with energies $\epsilon/2$. (Note that in calculating the elementary symmetric function the k index in cannot repeat itself.) Since

$$(1+x_1\lambda)^K = \sum_n \binom{K}{n} x_1^n \lambda^n \quad (4.8)$$

with

$$\binom{K}{n} = \frac{K!}{n!(K-n)!} \quad (4.9)$$

we get

$$Z_n^+ = \binom{K}{n} e^{-n\beta\epsilon/2}. \quad (4.10)$$

Similarly for the ‘‘lower-level’’ particles we get

$$Z_n^- = \binom{K}{n} e^{+n\beta\epsilon/2}. \quad (4.11)$$

Hence the N -particle partition function is given by

$$Z_N = \sum_{m+n=N} Z_m^+ Z_n^-. \quad (4.12)$$

To illustrate the dependence of the N -particle partition function on the variables x_1 and x_2 we calculate Z_N for the lowest values of N . Even though x_1 and x_2 are inverses of each other we will write their product explicitly to illustrate the underlying structure. One gets

$$Z_1 = K(x_1 + x_2), \quad (4.13)$$

$$Z_2 = \frac{K(K-1)}{2} (x_1^2 + x_2^2 + x_1 x_2) + \frac{K(K+1)}{2} (x_1 x_2), \quad (4.14)$$

$$Z_3 = \frac{K(K-1)(K-2)}{3!} (x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3) + \frac{K(K+1)(K-1)}{3} (x_1 x_2)(x_1 + x_2), \quad (4.15)$$

$$Z_4 = \frac{K(K-1)(K-2)(K-3)}{4!} (x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_2^4) + \frac{K(K+1)(K-1)(K-2)}{8} (x_1 x_2)(x_1^2 + x_2^2 + x_1 x_2) + \frac{K^2(K^2-1)}{12} (x_1^2 x_2^2), \quad (4.16)$$

and so on. In Eqs. (4.13) through (4.16) one notices the appearance of both complete and elementary symmetric functions of x_1 and x_2 .

There is a much faster way to calculate these partition functions. Noting that the operators

$$\hat{J}_0 = \frac{1}{2} \sum_{k=1}^K (f_{1,k}^\dagger f_{1,k} - f_{2,k}^\dagger f_{2,k}), \quad (4.17)$$

and

$$\hat{J}_+ = f_{1,k}^\dagger f_{2,k} = (\hat{J}_-)^\dagger, \quad (4.18)$$

generate an $SU(2)$ algebra one can write the Hamiltonian in Eq. (4.1) as an element of this algebra

$$\hat{H} = v\hat{J}_0 + \frac{t}{2}(\hat{J}_+ + \hat{J}_-). \quad (4.19)$$

As a result $\exp(-\beta\hat{H})$ is an element of the corresponding $SU(2)$ group and the partition function is the sum of traces (characters) of all possible representations of this group element. In the fundamental (two-dimensional or spinor) representation this group element takes the form

$$e^{-\beta\hat{H}} = \begin{pmatrix} e^{-\beta\epsilon/2} & 0 \\ 0 & e^{+\beta\epsilon/2} \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}. \quad (4.20)$$

We want to express the total partition function of Eq. (4.7) in terms of the characters of this $SU(2)$ algebra. Noting

$$(1+t)^K = \sum_n \binom{K}{n} t^n \quad (4.21)$$

and Eq. (B5) of the Appendix B we can write Eq. (4.7) as

$$Z = (1+x_1)^K (1+x_2)^K = \sum_{n_1=0}^K \sum_{n_2=0}^K \det \left[\binom{K}{n_j+i-j} \right] \chi_{(n_1, n_2)}(e^{-\beta\hat{H}}). \quad (4.22)$$

Using N particles one can construct those representations of the $SU(2)$ group in Eqs. (4.17) and (4.18) where $N = n_1 + n_2$. The easiest way to see that is to consider the grand canonical partition function

$$Z = (1+\lambda x_1)^K (1+\lambda x_2)^K = \sum_{n_1=0}^K \sum_{n_2=0}^K \det \left[\binom{K}{n_j+i-j} \right] \det[h_{n_j+i-j}(\lambda x_1, \lambda x_2)]. \quad (4.23)$$

Since the complete symmetric function satisfies the condition

$$h_n(\lambda x_1, \lambda x_2) = \lambda^n h_n(x_1, x_2), \quad (4.24)$$

one can write the character in Eq. (4.23) as

$$\det[h_{n_j+i-j}(\lambda x_1, \lambda x_2)] = \lambda^{n_1+n_2} \det[h_{n_j+i-j}(x_1, x_2)] \quad (4.25)$$

and the proof follows.

For $N=1$ using Eq. (B1) one has $\chi_{(1,0)}(e^{-\beta\hat{H}}) = h_1(x_1, x_2) = x_1 + x_2$ with the coefficient

$$\binom{K}{1} = K, \quad (4.26)$$

i.e., the result given in Eq. (4.13). For $N=2$, there are two possibilities: $n_1=2, n_2=0$ and $n_1=1, n_2=1$. To the expansion in Eq. (4.22) these contribute the terms

$$\binom{K}{2} h_2(x_1, x_2) + \det \begin{pmatrix} K & 1 \\ \frac{K(K-1)}{2} & K \end{pmatrix} \times [h_1^2(x_1, x_2) - h_2(x_1, x_2)], \quad (4.27)$$

which, after evaluating the determinants, gives Eq. (4.14). One can similarly calculate the coefficients of the $n_1+n_2=3$ and 4 terms in Eq. (4.22) to obtain Eqs. (4.15) and (4.16), respectively.

V. CONCLUSIONS

In this paper we showed that various partition functions for free (either noninteracting or those that interact through one-body Hamiltonians) particles can be written as a sum of one or more group characters. This result is not surprising since the partition function, being a trace, is invariant under the exchange of single-particle energies, hence it can be written in terms of either elementary or complete symmetric functions which form a complete basis for any function that is symmetric under the exchange of its variables. The resulting expressions are, however, very useful to simplify the calculations of the partition functions for particles that carry internal quantum numbers.

One should emphasize that our techniques, being combinatorial in nature, can be used to describe particle multiplicity distributions even in those situations where one does not start from a partition function or even when temperature is not well defined. Such applications range from pion multiplicity distributions in heavy-ion collisions [8] (where a temperature can be defined for a system) to fermion-pair production by a time-varying electric field [24,25] (where the concept of temperature is not introduced).

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APPENDIX A: SYMMETRIC FUNCTIONS

The complete homogeneous symmetric function, $h_n(x)$, of degree n in the arguments $x_i, i=1, \dots, N$, is defined as the sum of the products of the variables x_i , taking n of them

at a time. For three variables x_1, x_2, x_3 , the first few complete homogeneous symmetric functions are

$$\begin{aligned} h_1(x) &= x_1 + x_2 + x_3, \\ h_2(x) &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3, \\ h_3(x) &= \sum_i x_i^3 + \sum_{i \neq j} x_i^2 x_j + x_1x_2x_3. \end{aligned}$$

One can write the generating function for h_n as

$$\frac{1}{\prod_{i=1}^N (1 - x_i z)} = \sum_n h_n(x) z^n. \quad (A1)$$

If x_1, x_2, x_3 are the eigenvalues of a matrix B , the symmetric functions can be written in terms of traces of powers of B , e.g.,

$$\begin{aligned} h_1(x) &= \text{Tr } B, \\ h_2(x) &= \frac{1}{2} [\text{Tr } B^2 + (\text{Tr } B)^2], \end{aligned}$$

and so on.

The elementary symmetric functions, $a_n(x)$, are defined in a similar way except that no x_i can be repeated in any product. Again for three variables x_1, x_2, x_3 , the first few elementary symmetric functions are

$$\begin{aligned} a_1 &= h_1, \\ a_2 &= x_1x_2 + x_1x_3 + x_2x_3, \\ a_3 &= x_1x_2x_3. \end{aligned}$$

One takes $a_n=0$ if $n>N$ and $a_0=h_0=1$. The generating function for a_n is given by

$$\prod_{i=1}^N (1 - x_i z) = \sum_n (-1)^n a_n(x) z^n. \quad (A2)$$

Note that, since the generating functions in Eqs. (A1) and (A2) are inverses of each other one can write h_k in terms of $a_i, i=1, \dots, k$ and vice versa. If one takes $x_i, i=1, \dots, N$ to be eigenvalues of an $N \times N$ matrix A , then $a_N(x) = \det A$ and $a_{N+1}(x) = 0 = a_{N+2} = \dots$.

APPENDIX B: CHARACTER EXPANSION FORMULAS

In this appendix we summarize the character expansion formulas of Refs. [16,17] and [18]. The representations of the $U(N)$ group are labeled by a partition into N parts: (n_1, n_2, \dots, n_N) where $n_1 \geq n_2 \geq \dots \geq n_N$ (see, for example, Ref. [9]). We denote the eigenvalues of the group element U in the fundamental representation by t_1, t_2, \dots, t_N . The character (trace of the representation matrix) of the irreducible representation corresponding to the partition (n_1, n_2, \dots, n_N) of non-negative integers is given by

$$\chi_{(n_1, n_2, \dots, n_N)}(U) = \det(h_{n_j+i-j}), \quad (\text{B1})$$

where h_n is the complete symmetric function in the arguments t_1, \dots, t_N of degree n . (For a review of its properties see Appendix A.) In these equations the arguments of the determinants indicate the (ij) -th element of the matrix the determinant of which is calculated.

To obtain the character expansion of Ref. [17] consider the power series expansion

$$G(x, t) = \sum_n A_n(x) t^n, \quad (\text{B2})$$

where x stands for all the parameters needed to specify the coefficients A_n . We assume that this series is convergent for $|t|=1$. Given N different t 's which we take eigenvalues of the matrix U : t_1, \dots, t_N , we write down the character expansion [Eq. (2.17) of Ref. [17]] using N copies of Eq. (B2)

$$\left(\prod_{i=1}^N G(x, t_i) \right) = \sum_{m_1=0} \sum_{m_2=0} \sum_{m_{N-1}=0} \sum_{n_N} \det(A_{n_j+i-j}) \times (\det U)^{n_N} \chi_{(\ell_1, \ell_2, \dots, \ell_N)}(U), \quad (\text{B3})$$

where

$$m_j = n_j - n_{j+1}, \quad j = 1, \dots, N-1. \quad (\text{B4})$$

If the sum over n in the expression Eq. (B2) we started with is restricted to the non-negative values of n (i.e., $A_n=0$ when $n<0$), then n_N is non-negative and we can absorb the term $(\det U)^{n_N}$ into the character to obtain

$$\left(\prod_{i=1}^N G(x, t_i) \right) = \sum_{n_1=0} \sum_{n_2=0} \cdots \sum_{n_N=0} \det(A_{n_j+i-j}) \chi_{(n_1, n_2, \dots, n_N)}(U). \quad (\text{B5})$$

Note that the summation in Eq. (B5) is over all irreducible representations of $U(N)$, but in Eq. (B4) it is restricted to those representations where the number of boxes in the last row of the Young tableau is zero *and* an additional summation over n_N , which, in general, can take both positive and negative values. For further details see Ref. [17].

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